1. The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = ax^3 + 15x^2 - 39x + b$$

and a and b are constants.

Given

- the point (2, 10) lies on C
- the gradient of the curve at (2, 10) is –3

(a) (i) show that the value of a is –2

(ii) find the value of b.

(4)

(b) Hence show that C has no stationary points.

(3)

(c) Write $\mathrm{f}(x)$ in the form $(x - 4)\mathrm{Q}(x)$ where $\mathrm{Q}(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = \mathrm{f}(0.2x)$$

and the coordinates axes.

(2)

(a)

(i) $\frac{dy}{dx} = 3ax^2 + 30x - 39$

$\frac{dy}{dx} = -3$ when $x = 2$ $\Rightarrow$ $-3 = 3a(2)^2 + 30(2) - 39$ (1)

$12a = -24$

$a = -2$ (1)

(ii) As $f(2) = 10$

$\Rightarrow (-2)(2)^3 + 15(2)^2 - 39(2) + b = 10$ (1)

$-16 + 60 - 78 + b = 10$

$b = 44$ (1)

(b) $f'(x) = -6x^2 + 30x - 39$ (1)

$b^2 - 4ac \Rightarrow 30^2 - 4(-6)(-39) = -36 < 0$

(1)

Since $b^2 - 4ac < 0$, $f'(x) \neq 0$ so no stationary point exists. (1)

c) $f(x) = -2x^3 + 15x^2 - 39x + 44$

$f(x) = (x-4)\,Q(x)$

$$
\begin{array}{r|l}
 & -2x^2 + 7x - 11 \quad 1 \\
\hline
x-4 & -2x^3 + 15x^2 - 39x + 44 \\
 & -\;(-2x^3 + 8x^2) \\
\hline
 & 7x^2 - 39x \\
 & -\;(7x^2 - 28x) \\
\hline
 & -11x + 44 \\
 & -\;(-11x + 44) \\
\hline
\end{array}
$$

$f(x) = (x-4)(-2x^2 + 7x - 11)$ 1

d) when $x = 0$, $f(0) = f(0.2 \times 0) = 44$

$(0, 44)$

when $y = 0$

$(0.2x - 4)(-2 \times (0.2x)^2 + 7(0.2x) - 11) = 0$

$\Rightarrow 0.2x - 4 = 0$

$x = 20$

$(20, 0)$

1st term 2nd term

$f(0.2x) = (0.2x - 4)(-0.08x^2 + 1.4x - 11) = 0$

$x = 20$ is the only solution to $f(0.2x) = 0$ since 2nd term is < 0 when we put into $b^2 - 4ac \Rightarrow 1.4^2 - 4(-0.08)(-11) = -1.56$

f(0.2x) intersects at y-axis

f(0.2x) intersects x-axis

Point of intersection : $(0, 44)$ 1 and $(20, 0)$ 1

2. (a) Factorise completely $9x - x^3$

(2)

The curve C has equation

$$y = 9x - x^3$$

(b) Sketch C showing the coordinates of the points at which the curve cuts the x-axis.

(2)

The line l has equation $y = k$ where k is a constant.

Given that C and l intersect at 3 distinct points,

(c) find the range of values for k, writing your answer in set notation.

Solutions relying on calculator technology are not acceptable.

(3)

a) $9x - x^3 \equiv x(9 - x^2)$ (1)

$\equiv x(3+x)(3-x)$ (1)

b) $y = 9x - x^3$

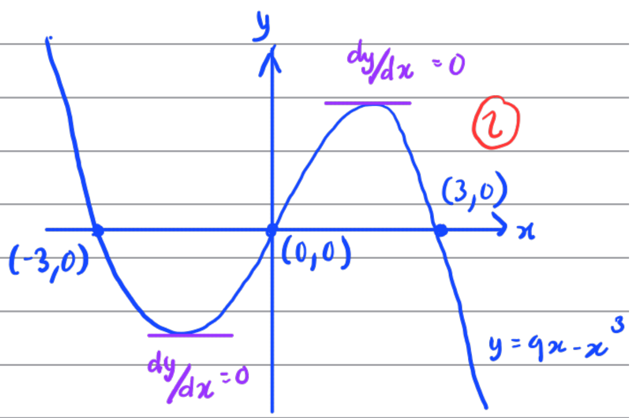
Curve C :

when $x = 0$, $y = 9(0) - (0)^3 = 0$

when $y = 0$, $0 = 9x - x^3$

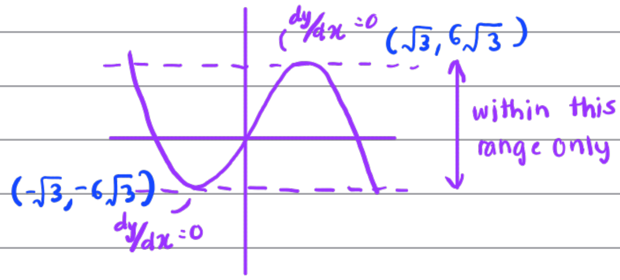
$0 = x(3+x)(3-x)$

$x = -3, 0, 3$



c) For line l to intersect at 3 points of curve C, the intersection points can only be within 2 turning points of the curve

Turning points : $\frac{dy}{dx} = 0$



$\frac{dy}{dx} = 9 - 3x^2$

$\therefore 9 - 3x^2 = 0$ (1)

$x^2 = 3$

$x = \pm\sqrt{3}$

when $x = \sqrt{3}$, $y = 9\sqrt{3} - (\sqrt{3})^3 = 6\sqrt{3}$ (1)

$x = -\sqrt{3}$, $y = 9(-\sqrt{3}) - (-\sqrt{3})^3 = -6\sqrt{3}$

$\therefore \{k \in \mathbb{R} : -6\sqrt{3} < k < 6\sqrt{3}\}$ (1)

3. Given that

$$y = 2x^2$$

use differentiation from first principles to show that

$$\frac{dy}{dx} = 4x$$

(3)

$y = 2x^2$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$ (1)

$= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 2x^2}{h}$

$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$

$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$ (1)

$= \lim_{h \to 0} 4x + 2h$

$\therefore$ when $h \to 0$,

$\frac{dy}{dx} = 4x$ (1)

4.

$$y = \sin x$$

where x is measured in radians.

Use differentiation from first principles to show that

$$\frac{dy}{dx} = \cos x$$

You may

- use without proof the formula for $\sin(A \pm B)$
- assume that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

let $f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \quad 1$$

$$1$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \quad 1$$

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

using $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$ as $h \to 0$

$$= \sin x(0) + \cos x(1) \quad 1$$

$$= \cos x$$

so $\frac{dy}{dx} = \cos x$ 1